Informal Remarks Preparatory For/ Complementary To Fei Yan's Talk

Simons Collaboration On Special Holonomy

Jan. 13, 2021

* Arbeitstagung

* Useful

Comments On:

1. Physics Background

2. Defects And Their BPS States

3. Class $\mathcal{S}$

4. Spectral Networks + Nonabelianization Map

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5. RH Problems, Integral Equations and Hyperkähler Geometry

(1) Physics Background

- Physicists assume many things and have intuitions and examples in mind that they take for granted, but which are not obvious to anyone else.

- Hamburg School On Higgs Bundles, Sept. 2018: Talk #84 on my homepage goes back to the beginning:

  * Branes + Geometrization of Higgs Mechanism
  * M5 Branes
  * 6d (2,0) Theory
  * Geometrical pictures for class S & their BPS states

Goal:

Explain the physics intuition behind theory of "Spectral networks"

② Defects

* $\exists$ rigorous theory in the context of extended TQFT ("cobordism hypothesis")

* In susy context: <u>New</u> BPS deg's

4d $\mathcal{N}=2$ SQFT: $\Omega(\gamma) \longleftrightarrow$ DT

line defects $\rightsquigarrow \underline{\overline{\Omega}}(L,\gamma)$ "framed"

surface defects $\rightsquigarrow \mu(\gamma_{ij})$ "soliton"

line defects in surface $\rightsquigarrow \underline{\overline{\Omega}}(L_{\wp},\gamma_{ij})$ "framed soliton"

* $\Omega(\gamma) \Rightarrow$ Well-known RH problem useful for construction of HK metrics on moduli

$\mathcal{M}$: space of solutions to Hitchin equations on a R.S. C (with singularities)

other BPS
deg's

* deg's $\Longrightarrow$ Similar RH problems

Constructing:

a.) Hyper-holo connections on certain vector bundles over $\mathcal{M}$

b.) Explicit construction of solutions to Hitchin equations on R. Surface C.

Now give some examples of defects.

- Example 1: Soliton & framed soliton deg's:

X exact Kähler $\omega = d\lambda$

✓ $W : X \longrightarrow \mathbb{C}$ superpotential (holo, Morse)

$(X, W) \longrightarrow$ 1+1 diml massive LG model. (Phys)

$\longrightarrow$ Fukaya-Seidel category (Math)

Critical points of W: $\{\phi_i\} \leftarrow$

$$\underline{\underline{\mathcal{X}_{ij}}} := \left\{ \phi : \underline{\mathbb{R}} \longrightarrow \underline{X} \,,\ \phi \begin{array}{l} \longrightarrow \phi_i \quad \underline{x \to -\infty} \\ \longrightarrow \phi_j \quad \underline{x \to +\infty} \end{array} \right\}$$

$$\underline{h} = \int_{\mathbb{R}} \Big[\phi^*(\lambda) - \mathrm{Re}\big(\underbrace{\zeta}_{\text{phase}} W(\phi(x))\big)\, dx \Big]$$

$\delta h = 0$ soliton eq. $\underline{\dfrac{d\phi}{dx} = \zeta \overline{\nabla W}}$

$$\boxed{\mu_{ij} = \chi\left(\text{Morse Complex}\right)}$$

"Soliton degeneracies"

Consider a manifold $\mathcal{C}$ of Morse superpotentials

$$W(\phi; z), \quad z \in \mathcal{C}$$

$\rightsquigarrow \wp(x): z_1 \rightsquigarrow z_2$ path in $\mathcal{C}$

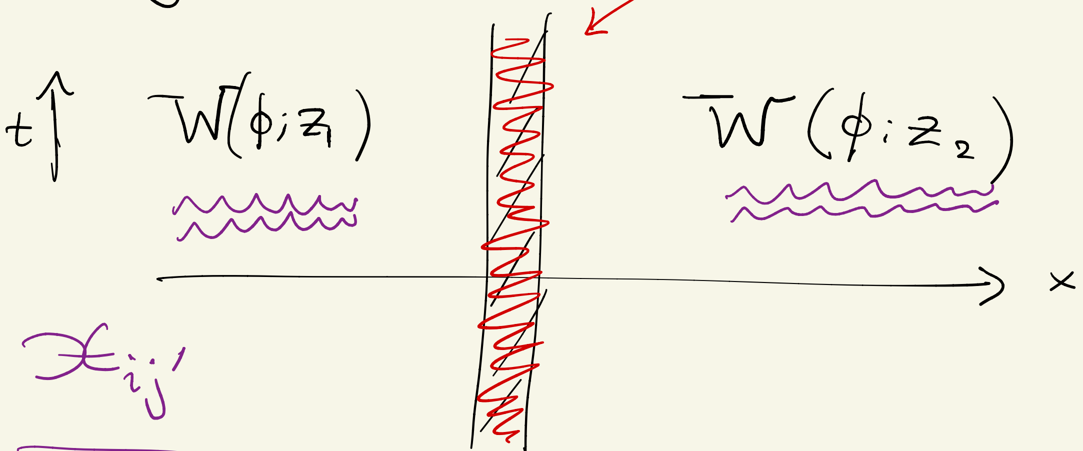
- $h = \int \left[\phi^*(\lambda) - \mathrm{Re}\left(\zeta W(\phi(x), z(x))\right) dx \right]$

$\delta h = 0$ soliton-like eq.

gives a new Morse complex.

Physical picture

defect: Line defect



$$\boxed{\mu_{ij'}(\wp) = \chi(\text{Morse Complex})} \quad \text{"framed BPS deg"}$$

• Example 2: Line Defects In 4d Gauge Theory w/ gauge group G

$$\boxed{\text{4d spacetime: } \quad \mathbb{R}^3 \times S^1}$$

$\varphi \in \mathfrak{g} \otimes \mathbb{C}$

Wilson:

$$L(R,\zeta) = \mathrm{Tr}_R \, P\exp \int_{\{\vec{x}_0\} \times S^1} \left(\zeta^{-1}\varphi + A + \zeta \varphi^{\dagger} \right)$$

$Q \in \Lambda_{\mathrm{char}}(G)$ highest wt of R.

't Hooft:

In path integral, put b.c.:

$$\vec{x} \to \vec{x}_0 \in \mathbb{R} \qquad F(\vec{x}) \sim P \sin\theta \, d\theta \, d\phi + \cdots$$

$$\Longrightarrow \mathrm{Re}(\zeta^{-1}\varphi) \sim \frac{P}{r} + \cdots$$

$$\boxed{P \in \Lambda_{\mathrm{cochar}}(G) \subset \mathfrak{g}}$$

Put them together

$P \oplus Q \in \Lambda_{char} \oplus \Lambda_{cochar}$

$\Rightarrow$ Wilson-'t Hooft lines: $L_{P,Q,\zeta}$

These are "UV descriptions of the line defects" because they tell us how to modify the path integral of the nonabelian field theory.

At $\vec{x} \longrightarrow \infty$ we have b.c.

- $\vec{F}(\vec{x}) \sim \gamma_m \sin\theta \, d\theta \, d\phi$ $\quad \gamma_m \in \Lambda^{cly}_{coroot}$

This is a long-distance/IR condition

- $\varphi \sim \langle \varphi \rangle = u \in \mathcal{B} =$ base of Hitchin fibration

Without line defects (smooth monopoles)

$$\Omega(\gamma_m, u) = \dim_{\mathbb{C}}\left(\mathrm{Ker}_{L^2} \not{D}_{\text{monopole mod. sp.}}\right)$$

$\dim(\text{Monopole Moduli}) \sim 4|\gamma_m|$

- With line defects:

Use singular monopoles w/ singularity P.

$F \longrightarrow P \sin\theta\, d\theta\, d\phi$
$\varphi \longrightarrow P/r$

$$\boxed{\underline{\overline{\Omega}}(L_{P,\zeta}, \gamma_m) = \dim_{\mathbb{C}}\left(\mathrm{ker}_{L^2} \not{D}_{\text{sing. mon. mod sp}}\right)}$$

The presence of the line defect at $\{\vec{x} = \vec{x}_0\} \times \mathbb{R}_t$ $(\times S^1_t)$ has modified the Hilbert space as a representation of $N=2$ super-Poinc. algebra (which contains Hamiltonian)

So spectrum of BPS – or groundstates – has changed.

- Example 3: Surface Defect.

General idea:

- 4d spacetime $\mathbb{R}^4$ or $\mathbb{R}^3 \times S^1_t$

  Coordinates (x, y, z, t)

- $\mathbb{R}^2_{y_0, z_0} \subset \mathbb{R}^4_{xyzt}$: Subspace with fixed y_0, z_0.

- 4d QFT $\mathcal{C}_4$ w/ gauge group G

- 2d QFT $\mathcal{C}_2$ w/ global symmetries G

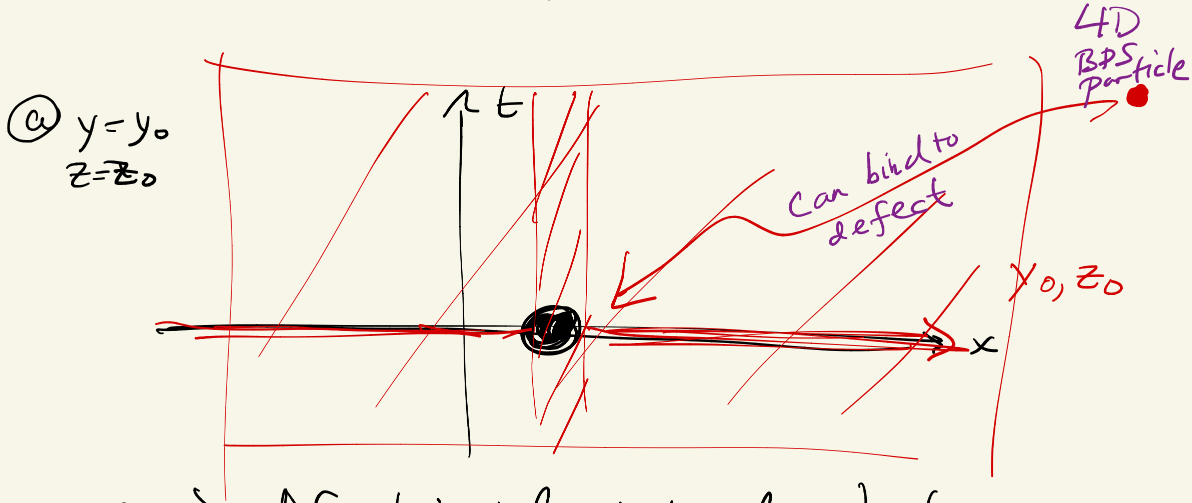
$\boxed{\text{2d-4d system:}}$

Couple $\mathcal{C}_4$ to $\mathcal{C}_2$ supported on $\mathbb{R}^2_{y_0 z_0}$

by adding to action

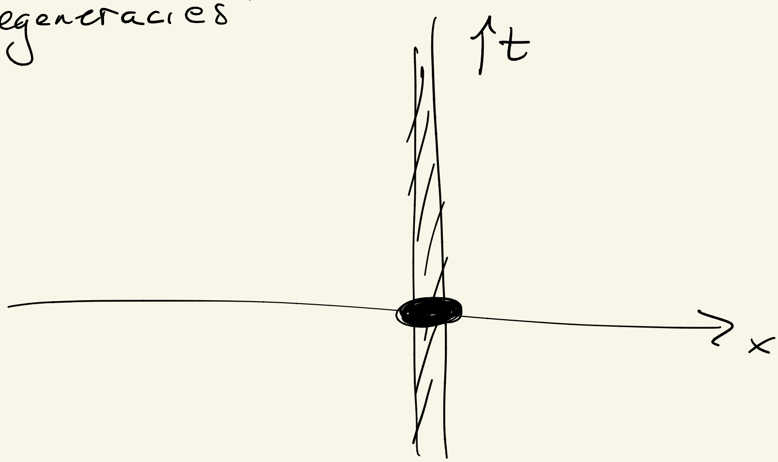
$$\int_{\mathbb{R}^2_{y_0 z_0}} \langle z^*(A_\mu), j^\mu \rangle \, d(\mathrm{vol})$$

The BPS particles of $\mathcal{C}_4$ can bind to the surface defect



$\rightsquigarrow$ Nontrivial interplay between 2d + 4d BPS degeneracies.

If we put a line defect in the surface defect we get "framed 2d4d degeneracies"



③ Class $\mathcal{S}$

$\mathfrak{g}$ = s.s. Lie algebra w/ ADE summands

$\Longrightarrow$ 6d QFT $\mathcal{S}(\mathfrak{g})$

See below for comment on the definition of $\mathcal{S}(\mathfrak{g})$

- C = Riemann surface
- D = "defect data"
  - divisor $\subset C$ support $\{P_\alpha\}$
  - choice of orbits in $\mathfrak{g}_{\mathbb{C}}$ at P_α

$\Longrightarrow$ 4d QFT $\mathcal{S}[\mathfrak{g}, C, D]$

Proof: $6 = 4 + 2$ $\quad M_4 \times C$

partial topological twist $\Rightarrow$ independence of some quantities on Kähler class of C. $\quad \text{area}(C) \to 0$

Example:

→ $\mathfrak{g}$: $su(2)$. 6d

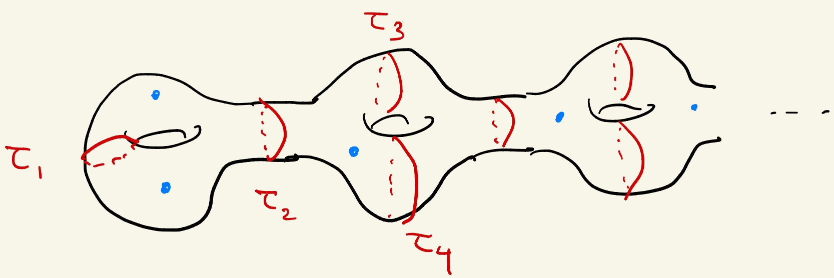
→ C : genus g

→ D : n punctures, orbit at p_α : $\begin{pmatrix} m_\alpha & \\ & -m_\alpha \end{pmatrix}$, $m_\alpha \neq 0$

4d gauge theory has gauge group G

- $\mathrm{Lie}(G) = su(2)^{\boxed{3g-3+n}} = \mathfrak{g}_{4d}$
- Coupling constants $\tau_i = \theta_i + \sqrt{-1}/g_i^2$ parametrize conformal structure of $\boxed{C_{g,n}}$

(Many different descriptions based on pants decomposition: Gaiotto)



$\bullet$ p_α

$n=0$ makes sense, but is qualitatively different.

There are 't Hooft-Wilson line defects $L_{P \oplus Q, \wp}$

$$P \oplus Q \in \Lambda_{cochar}(\mathfrak{g}_{4d}) \oplus \Lambda_{char}(\mathfrak{g}_{4d})$$

Drukker-Morrison-Okuda: These are Dehn-Thurston coordinates for isotopy class of closed 1-dimensional submanifold $\wp \subset C$

(at least .... when $\wp$ is connected....)

So we label line defects by

$$\boxed{\wp = \text{isotopy class of closed curve in } C}$$

Two "facts"

about 6d theory $S[\mathfrak{g}]$

N.B. No definition of $S[\mathfrak{g}]$ exists, even by physical standards where it is considered "obvious" that four-dimensional (nonanomalous) gauge theories exist.

An attempt to write a list of working rules ("axioms") which physicists use to produce mathematically well-defined statements and conjectures can be found in

my Felix Klein lecture notes in
Section 6.6 pp. 78–80.
See talk #47 on my homepage.

1. $S[\mathfrak{g}]$ has surface defects

In 6d spacetime $\mathbb{R}^3 \times S^1 \times C$

(A) $\mathrm{Supp}(\mathbb{S}) = \{\vec{x}_0\} \times S^1 \times \wp \qquad \wp \subset C$

$\Longrightarrow$ Line defect in 4d theory on $\{\vec{x}_0\} \times S^1$ $\quad \boxed{L_{\wp,\zeta,R}}$

Isotopy class of $\wp$ is a "UV label" generalizing the labels of 't Hooft - Wilson lines

(B) $\mathrm{Supp}(\mathbb{S}) = \boxed{\mathbb{R}_{y_0, z_0} \times S^1}_{\subset \mathbb{R}^3} \times \{z\}, \quad z \in C$

$\Longrightarrow$ Surface defect in 4d theory, $\mathbb{S}_z$

$$\boxed{\text{More careful analysis: } L_\wp \text{ also labeled by rep } R \text{ of } \mathfrak{g} \text{ and phase } \zeta.}$$

② When $S[\mathfrak{g}]$ is compactified on a circle, the LEET is

5D SYM $\Longrightarrow$

- $\mathfrak{g}$ gauge connection A
- $\mathfrak{g}_{\mathbb{C}}$ adjoint scalar φ

$QA = \psi$

$0 = Q\psi = F^+ + [\varphi, \varphi^\dagger]$

$= 0$

$$
\begin{tikzcd}
 & \boxed{S[\mathfrak{g}] \text{ on } \mathbb{R}^3 \times S^1_R \times C} \arrow[dl, "C"', "\mathrm{area}(C) \ll R^2"'] \arrow[dr, "S^1", "R^2 \ll \mathrm{area}(C)"] & \\
\boxed{S[\mathfrak{g}, C, D] \text{ on } \mathbb{R}^3 \times S^1} \arrow[d, "E \ll \frac{1}{R}"'] & & \boxed{\mathfrak{g} \text{ SYM on } \mathbb{R}^3 \times C} \arrow[d, "E \ll \frac{1}{\sqrt{\mathrm{area}(C)}}"] \\
\boxed{\text{HK } \sigma\text{-model}} \arrow[rr, phantom, "="] & & \text{HK } \sigma\text{-model} \\
\mathbb{R}^3 \to \mathcal{M}_{SW} & & \boxed{\mathbb{R}^3 \to \mathcal{M}_{\text{Hitchin}}}
\end{tikzcd}
$$

Answer to question: "How did you get the Hitchin equations"

10D SYM (others are reductions/trunc's)

$$A_M, \lambda \qquad M = 0, \dots, 9$$

$$Q\lambda = 0 \implies \gamma^{MN} F_{MN}\, \delta = 0.$$

for a suitable spinor δ

one example: $F^+ = 0$.

another example: Hitchin eqs.

We did not follow Hitchin's route of reducing $F^+ = 0$ to two dimensions.

IR description

$\langle \varphi \rangle = \lim \varphi(\vec{x})$

$\mathcal{B}$ = Coulomb branch = base of Hitchin fibration

$$\begin{array}{c} \tilde{C} \subset T^*C \\ \downarrow \pi \\ C \end{array}$$

spectral curve = Seiberg-Witten curve

Abelian gauge theory $U(1)^r$

$\boxed{\begin{array}{l} \lambda = \text{Liouville form} \\ \quad = \text{S-W differential} \end{array}}$

Electro-magnetic charge lattice of IR Ab. Thry

$\Rightarrow$ Γ is a subquotient of $H_1(\Sigma, \mathbb{Z})$

$Z = \oint \lambda : \Gamma \longrightarrow \mathbb{C}$ determines IR LEET

$$\Longrightarrow \boxed{\Omega(\gamma; u) \text{ etc.}}$$

T.B. talk $\mathfrak{g} = A_1$ $\Gamma = H_1^-(\Sigma, \mathbb{Z})$

④

Spectral Networks

&

Nonabelianization Map

But we also have other BPS degeneracies:

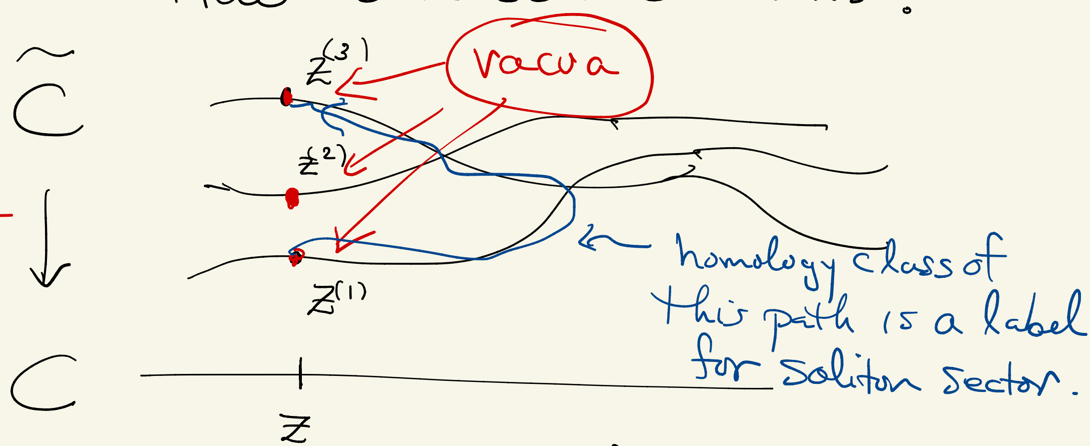
① $\boxed{\overline{\underline{\Omega}}(L_{\mathcal{R},\rho,\underline{\zeta}}\,;\underline{u})}$ (framed BPS degeneracies)

W.C. of $\overline{\underline{\Omega}}$ can be deduced by a simple physical argument and consistency then implies KS WCF for $\Omega(\gamma;u)$

2d "QFT" $\quad$ $\boxed{\phi_i \text{ of } W}$

② $\mathcal{D}_z$ has soliton degeneracies

How to label solitons?



$$\pi^{-1}(z) = \{ z^{(i)} \}$$

Soliton sectors of $\mathcal{S}_z$ are labeled by $\gamma \in \Gamma(z, z) := \bigcup_{i,j} \Gamma_{ij}(z, z)$ ← sheets

$$\Gamma_{ij}(z, z) = \left\{ \text{Chains } c \text{ in } \tilde{C} \text{ s.t. } \partial c = z^{(i)} - z^{(j)} \right\} / \sim$$

homology

Central Charges

$\partial \gamma = z^{(i)} - z^{(j)}$

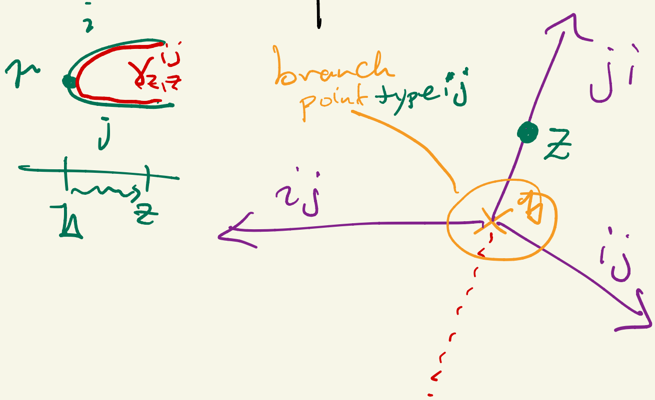
$$Z(\gamma) = \oint_\gamma \lambda$$

$\exists$ Soliton degeneracies: $\mu(\gamma)$

Physical defn of Spectral Network

$$\mathcal{W}_\zeta = \left\{ z \in C \;\middle|\; \begin{array}{l} \exists\, \gamma \in \Gamma(z, z) \\ Z(\gamma) = \zeta \cdot |Z(\gamma)| \\ \mu(\gamma) \neq 0 \end{array} \right\}$$

There is an algorithm for constructing $\underline{\mu(\gamma)}$ by evolving the spectral network from the branch points of $\pi : \tilde{C} \to C$

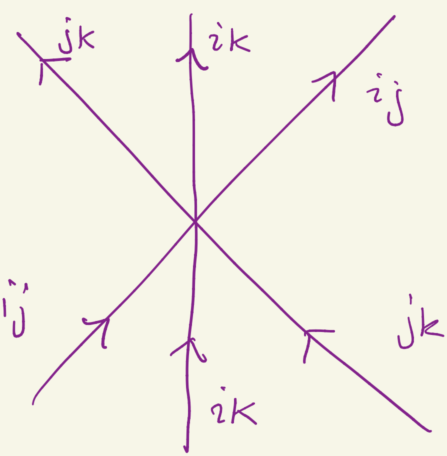


WKB curves

$$\text{phase}\left[\int_{z_1}^{z} \underline{\left(\lambda^{(i)} - \lambda^{(j)}\right)}\right] = \underline{\zeta}$$

$$\boxed{\mu\left(\gamma^{ij}_{z,z}\right) = 1}$$

Then apply simple wall-crossing formulae when lines cross



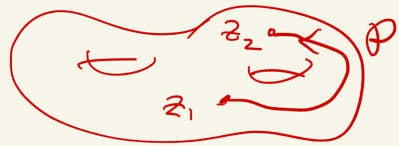
3 or more sheets

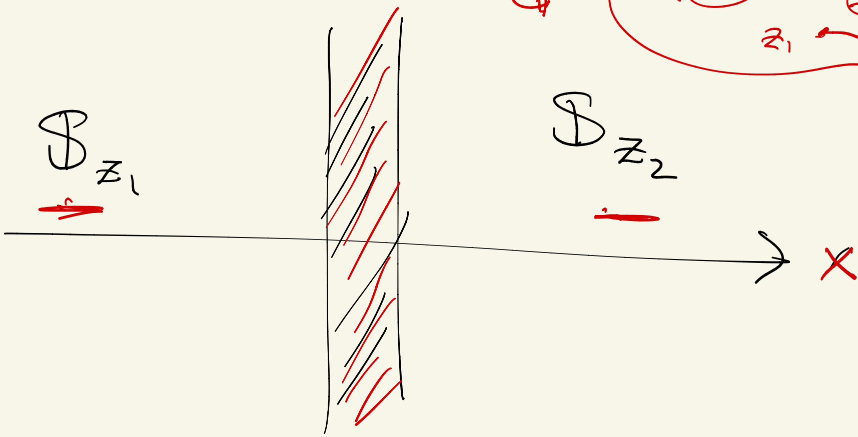
This can even be implemented by computer:

1. Andy Neitzke homepage
2. LOOM : software written by Pietro Longhi & Chan Park

③ We can also have domain walls between $\mathfrak{B}_{z_1}$ and $\mathfrak{B}_{z_2}$ for $z_1, z_2 \in \mathbb{C}$, just like the LG case

$\mathbb{C}$

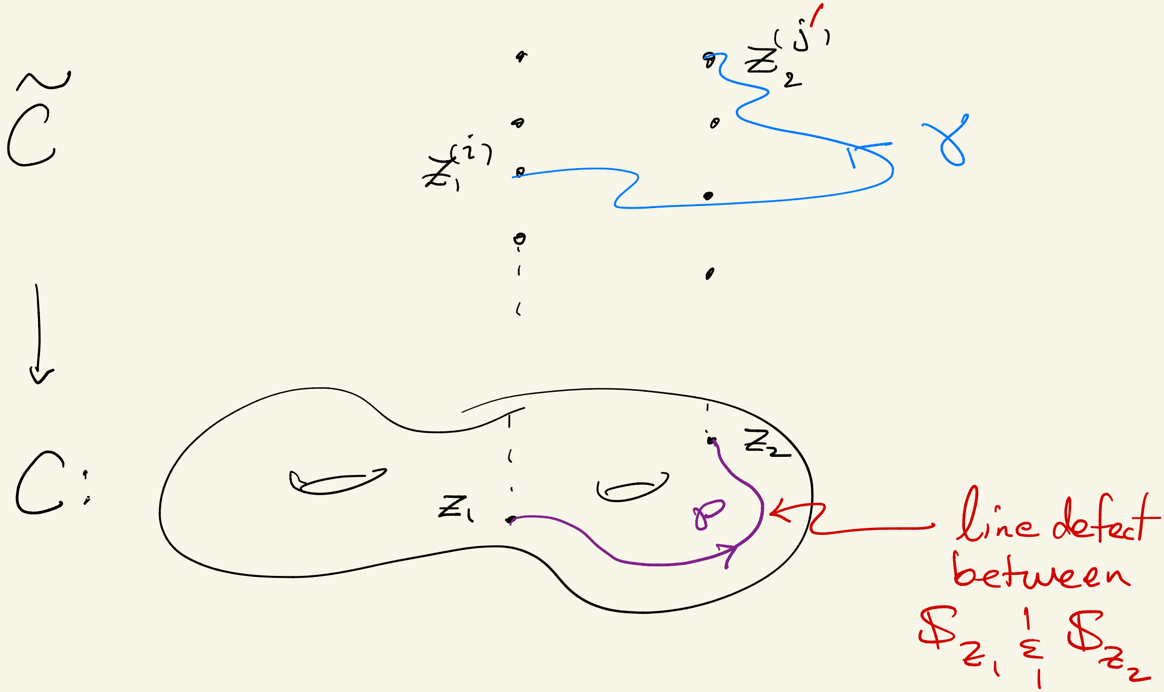




and these are labeled by isotopy classes of paths

$$\wp : z_1 \rightsquigarrow z_2 \quad \text{in } \mathbb{C}$$

$\Longrightarrow$ Framed BPS solitons



<u>Soliton sectors</u>:

$$\gamma \in \underline{\Gamma(z_1, z_2)} := \bigcup_{i,j'} \underline{\Gamma_{ij'}(z_1^{(i)}, z_2^{(j')})}$$

$$\Gamma(z_1^{(i)}, z_2^{(j')}) = \{ \text{Chains in } \tilde{C} \text{ with } \partial c = z_1^{(i)} - z_2^{(j')} \} / \sim$$

<u>Central charge</u> $\underline{Z(\gamma) = \int_\gamma \lambda}$

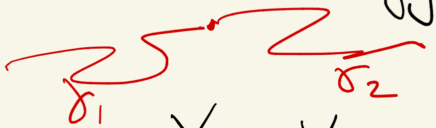
The BPS degeneracies $\overline{\underline{\Omega}}(\wp, \zeta, \gamma)$ are determined as follows:

→ Homology Path Algebra:

$\pi_{\leq 1}(\tilde{C})$

$\downarrow$

$H_{\leq 1}(\tilde{C})$



$$X_{\gamma_1} X_{\gamma_2} = \begin{cases} X_{\gamma_1+\gamma_2} & \text{if comp. makes sense} \\ 0 & \text{else} \end{cases}$$

$$\boxed{\mathbb{F}(\wp, \zeta) := \sum_{\gamma \in \Gamma(z_1, z_2)} \overline{\underline{\Omega}}(\wp, \zeta, \gamma) X_\gamma}$$

"Formal parallel transport"

$\wp : z_1 \rightsquigarrow z_2$ path in C

Claim [GMN, 2012]: $\exists!$ degeneracies

1.) $\overline{\Omega}(\rho, \zeta, \gamma)$ $\forall \rho$, $\forall \gamma \in \Gamma(z_1, z_2)$

2.) $\mu(\gamma)$ $\gamma \in \Gamma(z, z)$ $\forall z \in C$

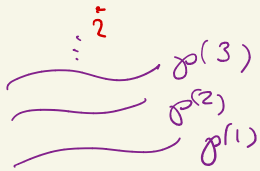
such that

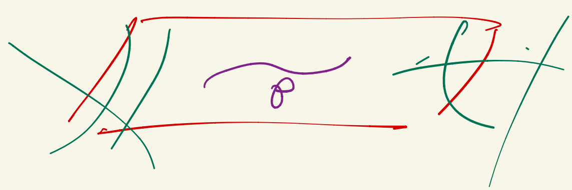
A.) Homotopy invariance: $\mathbb{F}(\rho_1, \zeta) = \mathbb{F}(\rho_2, \zeta)$
if $\rho_1 \sim \rho_2$ (fixed endpoints)

B.) Gluing: $\mathbb{F}(\rho_1, \zeta)\mathbb{F}(\rho_2, \zeta) = \mathbb{F}(\rho_1 \circ \rho_2, \zeta)$

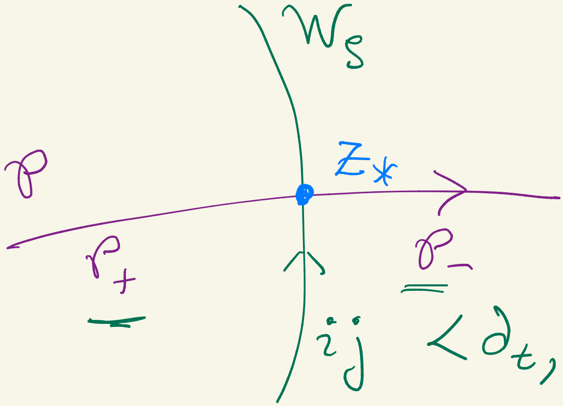
C.) If $\rho \cap W_\zeta = \phi$.

$$F(\rho, \zeta) = \sum_{\text{sheets}} X_{\rho^{(i)}} := D(\rho)$$





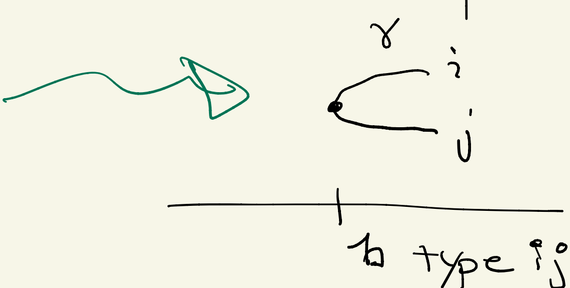
D.) Detour Rule



$\langle \partial_t, \lambda^{(i)} - \lambda^{(j)} \rangle = \zeta$

$$\boxed{\mathbb{F}(\rho, \zeta) = D(\rho_+) \prod_{\gamma \in \Gamma_{ij}(z_*, z_*)} (1 + \mu(\gamma) X_\gamma)\, D(\rho_-)}$$

Idea of Proof: Build up the $\mu(\gamma)$ and $\overline{\Omega}(\rho, \zeta, \gamma)$ starting from the branch points:



$\mu(\gamma) = 1$

Remarks

1.) Replacing $X_\gamma \Rightarrow$ $\boxed{\text{Parallel transport by flat } GL(1,\mathbb{C}) \text{ connection } \nabla^{ab} \text{ on } \tilde{C}}$

$$\mathbb{F}(p,\zeta) = \sum \overline{\underline{\Omega}}(p,\zeta,\gamma) \exp \int_\gamma \nabla^{ab}$$

is the $\parallel$-transport for a nonabelian flat connection on C

- gives the <u>nonabelianization map</u>

$$\Psi_{W_\zeta} : \mathcal{M}(\tilde{C}, GL(1)) \dashrightarrow \mathcal{M}_{flat}(C, GL(K))$$

$K = \#$ sheets

Claim: $\mathbb{F}$ is holomorphic symplectic.

Only provides coordinates on a chart in $\mathcal{M}_{flat}(C, GL(K))$ determined by W_ζ

2.) Fei & Andy — following some earlier work have sought to generalize the homology path algebra to a noncommutative Heisenberg algebra (for $\wp$ closed)

3.) The 2d theory $\mathcal{S}_z$ has an A_∞ Category of branes (generalizing the Fukaya-Seidel category)

There should be categorical analogs of the above where $\boxed{\mathbb{F}(\wp)}$ is a functorial analog of flat parallel transport.

$$\mathbb{F}(\wp) : \mathfrak{Br}(\mathcal{S}_{z_1}) \longrightarrow \mathfrak{Br}(\mathcal{S}_{z_2})$$

For the case of L.G. models this was actually constructed in Gaiotto-Moore-Witten, 2015.

It should lead to a categorified version of Stokes phenomenon.

S_{ij} walls were associated with ("S-wall") crossing functors $\mathfrak{S}_{ij}$ in GMW. These are categorical generalizations of Stokes factors.

Example of 2d4d

4d SU(2) pure.

$C = \mathbb{P}^1$ with 2 punctures

$$\lambda^2 = \left(\frac{\Lambda^2}{z^3} + \frac{u}{z^2} + \frac{\Lambda^2}{z} \right) dz^{\otimes 2}$$

Couple this to $\mathbb{CP}^1$ model. It has global SU(2) symmetry.

for z in certain regions

Can identify w/ $\mathcal{S}_z$!

See GMN 1103.2598 section 8.3

Two math references related to this:

- A_∞ category of branes for $\mathfrak{B}_z$:

  G. Kerr + Y. Soibelman 1711.03695

- Math formulation/construction of functor $\mathbb{F}(\wp)$ giving categorical Stokes factors:

  M. Kapranov, Y. Soibelman, L. Soukhanov, 2011.00845